$\boxed{\text{Operadic renormalisation group}}$ [Higher structures in Renormalisation, "Vienna"]

Joint work (to appear) with Ricardo Campos

1. BV vs HTT
2. Deformation theory of homotopy algebras
3. Applications

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1. Batalin-Vilkovisky formalism = Homotopy transfer theorem

BV formalism:

original space $\searrow$; $\swarrow$ sub space

$$C \hookleftarrow H$$

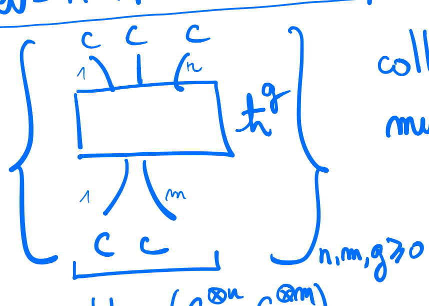
action S $\rightarrow C$; $S \dashrightarrow$ effective action S^{eff} $\rightarrow H$: formulas = Feynman diagrams

$S \in S(C \oplus sC^*)[[\hbar]]$: quantum Weyl algebra

$\hookrightarrow$ solution to the quantum master equation (aka Maurer-Cartan eq.)

$$\boxed{\hbar \Delta S + \frac{1}{2}[S,S] = 0} \iff \Delta e^{S/\hbar} = 0$$

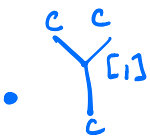
Losev-Mnev-Merkulov interpretation

$S \longleftrightarrow$  $\in \mathrm{Hom}(C^{\otimes n}, C^{\otimes m})$

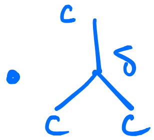
collection of operations with multiple inputs and outputs

MC equation $\iff$ homotopy unimodular Lie bialgebra

i.e an "∞" relaxed version of



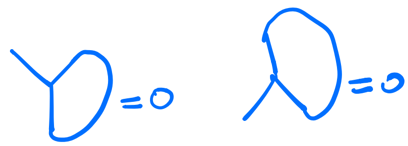
Lie bracket (Jacobi relation)



Lie cobracket ("co" Jacobi relation)

compatibility relation

compatibility with the trace



C : chain complex H : its homology groups

contraction $\boxed{h} \circlearrowright C \underset{i}{\overset{p}{\rightleftarrows}} H$

contracting homotopy

||

$\boxed{\text{Propagator}}$

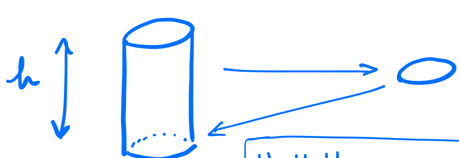
S = (homotopy) unimodular Lie bialgebra

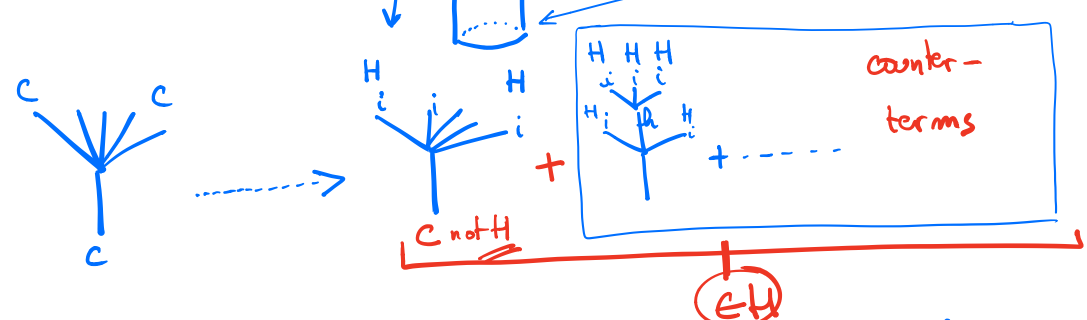
S^{eff} = homotopy transferred structure

formula = sum of labelled graphs

<u>Homotopy transfer theorem</u> [Hoffbeck-Leray-V.]

Contraction $h \circlearrowright C \underset{i}{\overset{p}{\rightleftarrows}} H$





⟷ Perturbation theory, Renormalisation theory

__Idea__: HTT gives more

$$\begin{tikzcd} C \arrow[r, squiggly, shift left] & H \arrow[l, squiggly, shift left, "\infty\text{-morphisms}"] \end{tikzcd}$$

$$\Downarrow$$

$$S \overset{\text{compare}}{\longleftrightarrow} S^{\text{transferred}}$$

__Ex:__ modules over the algebra of dual numbers $T(x)/(x^2)$,

S^{tr} = Connes' B $\Rightarrow$ definition of cyclic homology

and $\underset{\infty\text{-morphisms}}{\leftsquigarrow}$ = Chern characters.

__Leading question:__ Interpretation of $\underset{\infty\text{-morphisms}}{\rightleftarrows}$ in the BV-formalism?

② Deformation theory of homotopy algebras (where the HTT sits)

Type of operations End_V	Governing notion $\mathcal{P}$	Type of "algebra" $\mathcal{P}\to\mathrm{End}_V$	Convolution algebra $\mathfrak{g}_{\mathcal{P},V} := (\mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{¡};\mathrm{End}_V),\star)$
$\mathrm{Hom}(V,V)$	assoc alg.	$\mathcal{P}$-module	associative
$\mathrm{Hom}(V^{\otimes n},V)$	operad	$\mathcal{P}$-algebra	pre-Lie
$\mathrm{Hom}(V^{\otimes n},V^{\otimes m})$	properad	$\mathcal{P}$-gebra	Lie admissible

$\mathcal{P}^{¡}$: Koszul dual co(pr) operad

$f, g : \mathcal{P}^{¡} \to \mathrm{End}_V$: $\quad f\star g : \underset{\in \mathcal{P}^{¡}}{\cdot} \xrightarrow{\Delta_{(1)}} \underset{\in \mathcal{P}^{¡}\otimes\mathcal{P}^{¡}}{\sum} \xrightarrow{f,\,g} \underset{\in \mathrm{End}_V\otimes\mathrm{End}_V}{\sum} \xrightarrow{\gamma} \underset{\in \mathrm{End}_V}{\cdot}$

__Def__ [Pre-Lie] $\boxed{(x\star y)\star z - x\star(y\star z) = (x\star z)\star y - x\star(z\star y)}$

$$\Downarrow$$

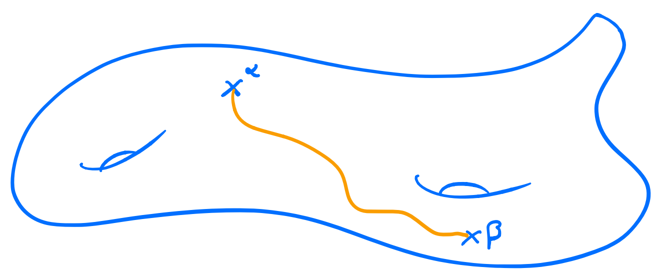
$[x,y] := x\star y - y\star x$ Lie bracket (Jacobi relation)

Def [Lie admissible] $[\,,\,]$ Lie bracket
$\Longleftrightarrow \star$ satisfies a 12-terms relation

α solution to the Maurer-Cartan equation $\partial\alpha + \frac{1}{2}[\alpha,\alpha] = 0$
in $\mathfrak{g}_{\mathcal{P},V}$ $\xleftrightarrow{1-1}$ $\mathcal{P}_\infty$-(al)gebra structure on V.

Ex: A_∞, L_∞, BiLie_∞, ...

MC($\mathfrak{g}$)



Integration: Gauge group $\boxed{G := (\mathfrak{g}_0, \mathrm{BCH}, 0)}$

↑ Baker-Campbell-Hausdorff formula

action

$\mathrm{MC}(\mathfrak{g})$

$\lambda.\alpha = \beta$
$\Updownarrow$
$\alpha \rightsquigarrow \beta$
∞-isomorphism

Issue: Both BCH and action formulas are complicated! (for me)

"Effective" Integration:

$$\mathfrak{g}_{\mathcal{P},V} = \mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{¡}, \mathrm{End}_V) \cong \underset{\substack{\cup \\ 1 : I \mapsto \mathrm{id}_V}}{\mathrm{Hom}_{\mathbb{S}}(I, \mathrm{End}_V)} \oplus \underbrace{\mathrm{Hom}_{\mathbb{S}}(\bar{\mathcal{P}}^{¡}, \mathrm{End}_V)}_{\bar{\mathfrak{g}}_{\mathcal{P},V} :=} \quad \& \quad \left\{\begin{matrix}\mathrm{MC}(\mathfrak{g}) \\ G\end{matrix}\right. \subset \bar{\mathfrak{g}}$$

Case $\boxed{1}$: $\mathcal{P}$ associative algebra $\Rightarrow (\bar{\mathfrak{g}}, \star)$ associative algebra

Exponential map: $e^{x} := \mathbb{1} + x + \frac{1}{2} x \star x + \frac{1}{6} x \star x \star x + \cdots$

Proposition

• $\bar{\mathfrak{g}} \underset{\exp}{\overset{\log}{\rightleftarrows}} \mathbb{1} \oplus \bar{\mathfrak{g}}$ inverse group isomorphisms

$G = (\bar{\mathfrak{g}}_0, BCH, 0)$ $\qquad \mathcal{G} := (\mathbb{1} \oplus \bar{\mathfrak{g}}_0, \star, \mathbb{1})$: Deformation gauge group

• Action of $\mathcal{G}$ on $MC(\mathfrak{g})$: $\lambda . \alpha = e^{\lambda} \star \alpha \star e^{-\lambda}$

Case $\boxed{2}$: $\mathcal{P}$ operad $\Longrightarrow (\bar{\mathfrak{g}}, \star)$ pre Lie algebra

Exponential map: $e^{x} = \mathbb{1} + x + \frac{1}{2} x \star x + \frac{1}{6} (x \star x) \star x + \cdots$

[Agrachev - Gamkrelidze]

Cayley?

Theorem [Dotsenko-Shadrin-V.]

• $\bar{\mathfrak{g}} \underset{\exp}{\overset{\log = \text{Magnus}}{\rightleftarrows}} \mathbb{1} \oplus \bar{\mathfrak{g}}$ inverse group isomorphisms

"group of formal flows"

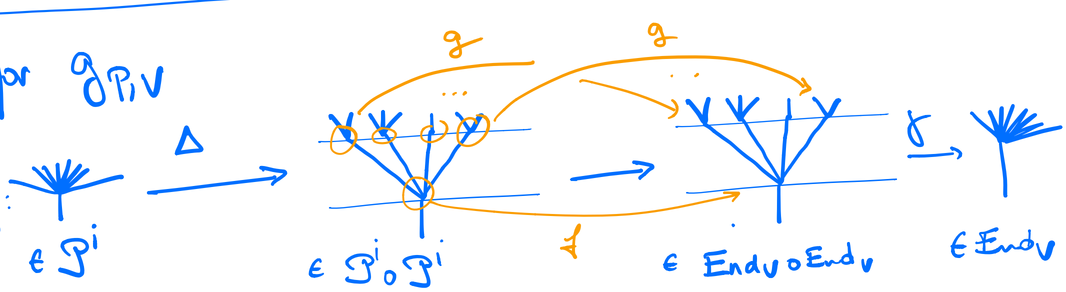
$G = (\bar{\mathfrak{g}}_0, BCH, 0)$ $\qquad \mathcal{G} := (\mathbb{1} + \bar{\mathfrak{g}}_0, \circledcirc, \mathbb{1})$

$\circledcirc$ = sum of symmetric braces

• Action of $\mathcal{G}$ on $MC(\bar{\mathfrak{g}})$: $\lambda . \alpha = (e^{\lambda} \star \alpha) \circledcirc e^{-\lambda}$

Rk: for $\mathfrak{g}_{\mathcal{P}, V}$

$f \circledcirc g$:



└ "very" operadic

Case $\boxed{3}$: P properad $\Longrightarrow (\bar{g}, \star)$ Lie admissible

? Exponential map: $e^x := \mathbb{1} + x + \frac{1}{2} x \star x + \frac{1}{6}\left(x \star (x \star x)\right) + \cdots$

does not work

does not work

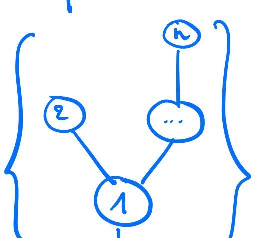
???

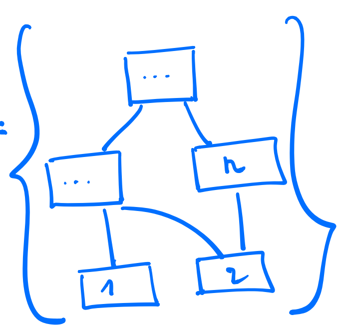
Algebraic structure on $\bar{g}_{P,V}$			
Minimal (infinitesimal)		Maximal (global)	Exponential
As	$=$	As	$e^x = \sum_{\mu \in As(n)} \frac{1}{n!} \mu(x, \dots, x)$
pre Lie	$\cong$	Rooted Trees (RT)	$e^x = \sum_{\mu \in RT(n)} \frac{1}{n!} \mu(x, \dots, x)$
Lie admissible	$\subsetneq$	Directed Simple Graphs (dsGra)	$e^x := \sum_{\mu \in dsGra(n)} \frac{1}{n!} \mu(x_1, \dots, x)$

↑ for the MC equation

↑ all operations

$\leadsto$ exp is no more an iteration of generating products.

$RT(n) =$ 

$ds\,Gra(n) =$ 

connected

at most one edge between 2 vertices

Theorem [Campos–V.] $\mathfrak{g}$: dsGra-algebra (like $\mathfrak{g}_{P,V}$)

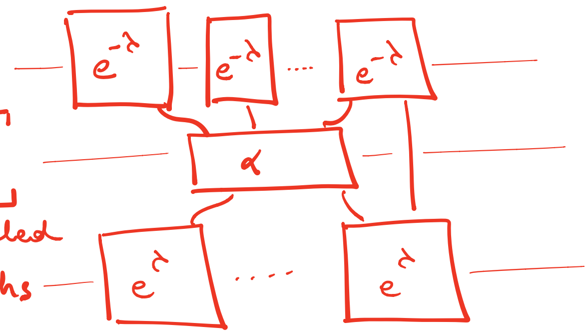
• $\bar{\mathfrak{g}} \underset{\exp}{\overset{\log = \text{new Magnus}}{\rightleftarrows}} \mathbb{1}\oplus\bar{\mathfrak{g}}$ inverse group isomorphisms

$G = (\bar{\mathfrak{g}}_0, BCH, 0)$ $\mathcal{G} := (\mathbb{1}+\bar{\mathfrak{g}}_0, \circledcirc, \mathbb{1})$

$$(\mathbb{1}+x)\circledcirc(\mathbb{1}+y) = \mathbb{1} + \sum_{\substack{\text{2-leveled} \\ \text{ds graphs } \gamma}} \frac{1}{|\mathrm{Aut}(\gamma)|}\,\gamma(x,y)$$

• $\mathcal{G}$ action on $MC(\bar{\mathfrak{g}})$:

$$\lambda \cdot \alpha = \sum_{\substack{\text{3-leveled} \\ \text{ds graphs}}}$$



3 Applications

$\exists$ 3 functorial ways to produce $\mathcal{P}_\infty$-gebras $\Leftarrow$ they all come from the above (deformation) gauge group action.

(i) → Twisting procedure

paradigm: V $\mathcal{L}_\infty$-algebra $\xrightarrow[\alpha \in MC(V)]{}$ V^α: "twisted" $\mathcal{L}_\infty$-algebra

Here: works in full generality for $\mathcal{P}_\infty$-gebras ($e^\alpha \in \mathcal{G}$)

↳ Ex: for homotopy involutive Lie bialgebras: Cieliebak–Fukaya–Latschev in symplectic field theories.

(ii) → <u>Koszul hierarchy</u>

<u>General case</u>: V chain complex $\longrightarrow$ $\mathcal{P}^{!}_{\infty}$-gebra structure on V

$\mathcal{P}$-gebra structure on V $\in \mathcal{G}$

<u>Ex</u>: $\mathcal{P} = As, Com$: formulas for the cumulants in NC probability à la Voiculescu.

(iii) → <u>Homotopy Transfer Theorem</u>

contraction: $h \circlearrowright C \underset{i}{\overset{p}{\rightleftarrows}} H$

$\uparrow$

$\alpha \in MC(\mathfrak{g}_{\mathcal{P},V})$

$(\alpha, h) \longrightarrow$ Fixed point equation $\longrightarrow$ solution $\Phi \in \mathcal{G}$

$\boxed{\text{Thm}}$ [Campos-V.]

$$\boxed{\alpha^{\text{transferred}} = \Phi . \alpha}$$

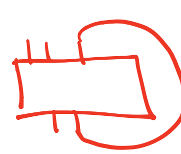
$\alpha \overset{\Phi}{\rightsquigarrow} \alpha^{tr}$
<u>∞-isomorphism</u>

action $\longrightarrow$ effective action

$\mathcal{G}$: "Renormalisation group"

<u>Finished?</u> Do we have enough higher algebra to deal with the BV formalism?

NOT YET : properads ----→ <u>wheeled</u> properads



<u>trace</u>

Thank you for your attention

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